3E Poi

Contents

3E Poisson Distribution

[ToC]

E.1 Poisson

$X \sim Poi(n, m, N)$ **Analogy:** events with rate λ per unit time.

$$\text{pmf}: \quad p(x) = p(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for} \quad x = 0, 1, 2, \ldots$$

$$\text{CDF}: \quad F(x) = P(X \leq x) = \sum_{k=0}^{x} p(x)$$

$$\text{mean}: \quad E(X) = \lambda$$

$$\text{var}: \quad V(X) = \lambda$$

$$\text{MGF}: \quad M(t) = \exp\{\lambda(e^t - 1)\}$$

```
dpois(2, lambda)        #pmf at x=2
ppois(2, lambda)        #CDF at x=2
qpois(.5, lambda)       #Inv CDF at q=.5
rpois(1000, lambda)      # random sample of size 1000
```

E.2 Poisson as a limit of Binomial

Poisson distribution is the limit of binomial distribution when $n \to \infty$, $p \to 0$, in such a way that $np \to \lambda$.

Starting from Binomial pmf and replacing $p = \lambda/n$,

$$p_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\begin{aligned}
&= \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \\
&= \frac{1}{x!} \frac{n!}{(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x} \\
&= \frac{1}{x!} \left(\frac{n!}{(n-x)!n^x}\right) \lambda^x \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x}
\end{aligned}$$

If we take the lim,

$$\lim_{n\to\infty} p_X(x) = P(X = x) = \lim_{n\to\infty} \frac{1}{x!}\left(\frac{n!}{(n-x)!n^x}\right)\lambda^x\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$
$$= \frac{\lambda^x}{x!}e^{-\lambda}$$

E.3 When Time Units are Changed

E.4 Example: Number of Tornados

Suppose the number X of tornadoes observed in a particular region during a 1-year period has a Poisson distribution with $\lambda = 8$.

1. What is the probability we get fewer than 4 tornados next year?

2. What is the probability we get fewer than 6 tornados in next two years?

E.5 Example: Aircraft arrivals

Suppose small aircraft arrive at a certain airport according to a Poisson process with rate $\alpha = 8$ per hour, so that the number of arrivals during a time period of t hours is a Poisson r.v. with $\lambda = 8t$.

1. What is the probability that exactly 6 small aircraft arrive during 1-hour period?

2. What are the expected value and standard deviation of the number of small aircraft that arrive during a 90-min period?

3. What is the probability that at least 20 small aircraft arrive during 3 hour period?