3E Poisson

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3E Poisson Distribution

[ToC]

E.1 Poisson

$X \sim Poi(n, m, N)$ **Analogy:** events with rate λ per unit time.

$$\text{pmf}: \quad p(x) = p(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for} \quad x = 0, 1, 2, \ldots$$

$$\text{CDF}: \quad F(x) = P(X \leq x) = \sum_{k=0}^{x} p(x)$$

$$\text{mean}: \quad E(X) = \lambda$$

$$\text{var}: \quad V(X) = \lambda$$

$$\text{MGF}: \quad M(t) = \exp\{\lambda(e^t - 1)\}$$

```
dpois(2, lambda)        #pmf at x=2
ppois(2, lambda)        #CDF at x=2
qpois(.5, lambda)       #Inv CDF at q=.5
rpois(1000, lambda)      # random sample of size 1000
```

E.2 Poisson as a limit of Binomial

Poisson distribution is the limit of binomial distribution when $n \to \infty$, $p \to 0$, in such a way that $np \to \lambda$.

Starting from Binomial pmf and replacing $p = \lambda/n$,

$$p_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{1}{x!} \frac{n!}{(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x}$$

$$= \frac{1}{x!} \left(\frac{n!}{(n-x)! n^x}\right) \lambda^x \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x}$$

If we take the lim,

$$\lim_{n\to\infty} p_X(x) = P(X = x) = \lim_{n\to\infty} \frac{1}{x!}\left(\frac{n!}{(n-x)!n^x}\right)\lambda^x\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$
$$= \frac{\lambda^x}{x!}e^{-\lambda}$$

E.3 When Time Units are Changed

E.4 Ex: Number of Tornados

Suppose the number X of tornadoes observed in a particular region during a 1-year period has a Poisson distribution with $\lambda = 8$.

1. What is the probability we get fewer than 4 tornados next year?

2. What is the probability we get fewer than 6 tornados in next two years?

E.5 Ex: Aircraft arrivals

Suppose small aircraft arrive at a certain airport according to a Poisson process with rate $\alpha = 8$ per hour, so that the number of arrivals during a time period of t hours is a Poisson r.v. with $\lambda = 8t$.

1. What is the probability that exactly 6 small aircraft arrive during 1-hour period?

2. What are the expected value and standard deviation of the number of small aircraft that arrive during a 90-min period?

3. What is the probability that at least 20 small aircraft arrive during 3 hour period?

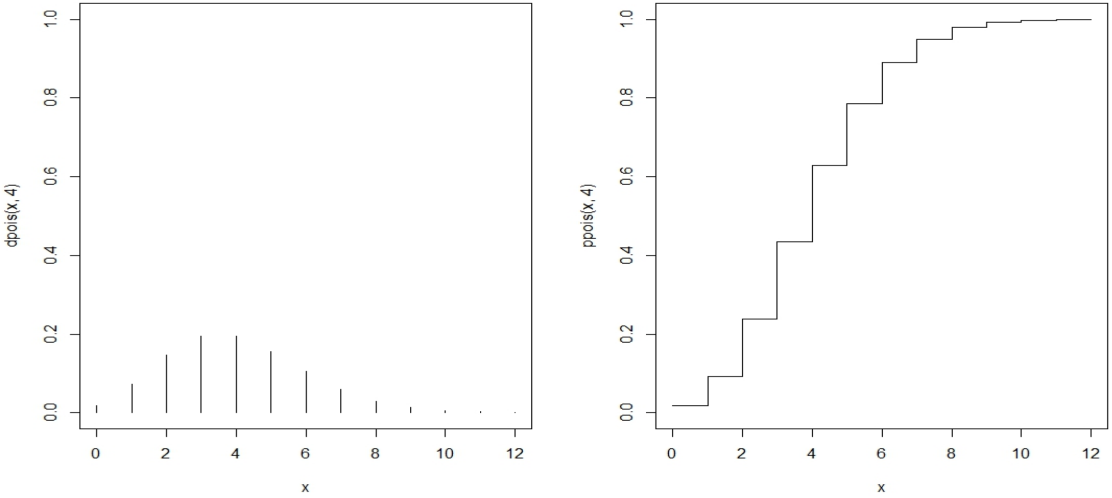
E.6 R code for Poisson(λ)

$x =$[number of events in a unit time] $\lambda =$[average number of events per unit time]

```
dpois(3,4)         #- p(3):  pmf of Poi(lambda=.5) at x=3
ppois(3,4)         #- F(3):  CDF of Poi(lambda=.5) at x=3


layout( matrix(1:2, 1, 2) )  #- Make plot layout side by side

x <- 0:12
plot(x, dpois(x, 4), type="h", ylim=c(0,1))   #- PMF plot -
plot(x, ppois(x, 4), type="s", ylim=c(0,1))   #- CDF plot -
```


dpois(x, 4)
ppois(x, 4)
x
x

E.7 Poisson process

- Let $N(t)$ denote the number of events before time t

$$P(N(t_2) - N(t_1) = x) = \frac{e^{-\lambda(t_2 - t_1)} \left[\lambda \left(t_2 - t_1\right)\right]^x}{x!}.$$

- Assume independence over disjoint time interval.

- Then waiting time between events will be iid Exponential with mean $1/\lambda$.

Poisson as a limit If we let $n \to \infty$, $p \to 0$, in such a way that $np \to \lambda$, then the pmf

$$\text{Binomial}(n, p) \to \text{Poisson}(x; \lambda).$$

E.8 Sum of Poisson is Poisson

- mgf for poisson

$$\begin{aligned} M_{X_1}(t) = E(e^{tX}) &= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(e^t\lambda)^x}{x!} \\ &= \frac{e^{-\lambda}\sum_{x=0}^{\infty}(e^t\lambda)^x}{x!} \\ &= e^{-\lambda}e^{(\lambda e^t)} = e^{\lambda(e^t-1)} \end{aligned}$$

- If $X_1, X_2 \sim$Poi(λ) and independent, since

$$M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_1}(t) = e^{\lambda(e^t-1)}e^{\lambda(e^t-1)} = e^{2\lambda(e^t-1)},$$

we see taht $X_1 + X_2 \sim$Poi(2λ).